Complementi sulla risposta nel dominio del tempo dei sistemi di primo e secondo ordine

Poles and Zeros of a First-Order System: An Example

Given the transfer function $G(s)$ in Figure 4.1(a), a pole exists at $s = -5$ and a zero exists at -2. These values are plotted on the complex s-plane in Figure 4.1(b) using an × for the pole and a ○ for the zero. To show the properties of the poles and zeros, let us find the unit step response of the system. Multiplying the transfer function of Figure 4.1(a) by a step function yields

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5} \tag{4.1}$$

where

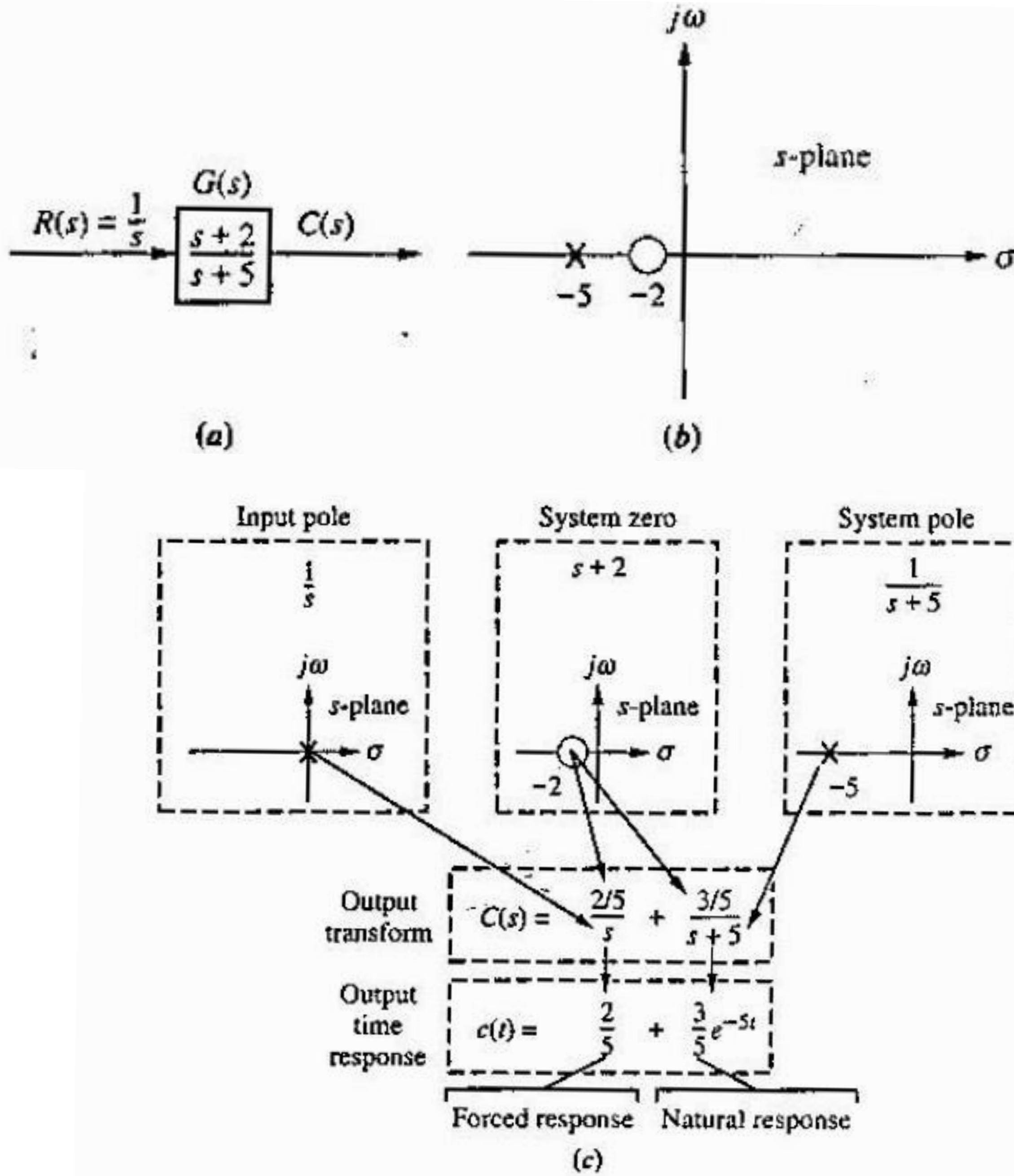
$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s \to 0} = \frac{2}{5}$$

$$B = \left.\frac{(s+2)}{s}\right|_{s \to -5} = \frac{3}{5}$$

Thus,

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t} \tag{4.2}$$

Figure 4.1
a. System showing input and output;
b. pole-zero plot of the system;
c. evolution of a system response. Follow blue arrows to see the evolution of the response component generated by the pole or zero.



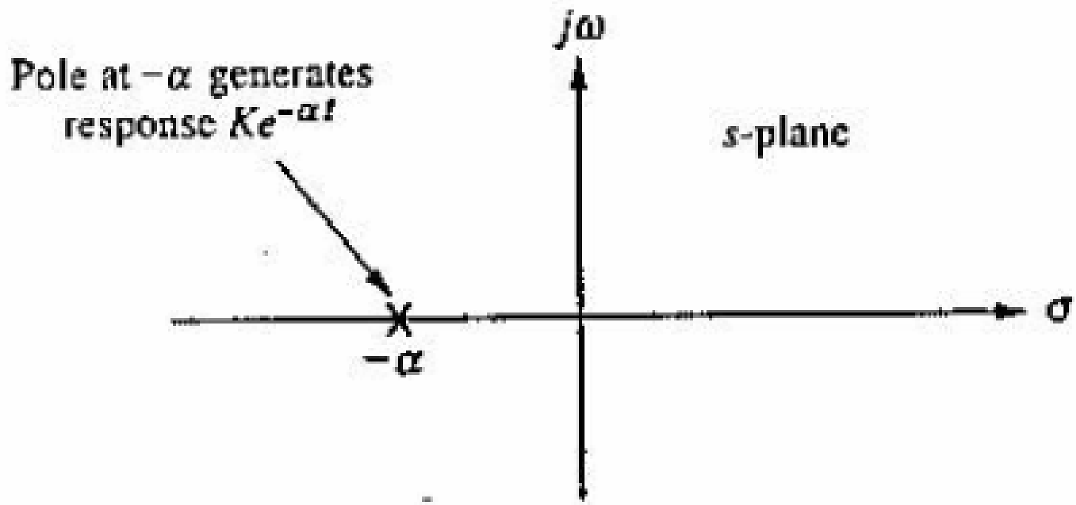
From the development summarized in Figure 4.1(*c*), we draw the following conclusions:

1. A pole of the input function generates the form of the *forced* response (i.e., the pole at the origin generated a step function at the output).

2. A pole of the transfer function generates the form of the *natural* response (i.e., the pole at -5 generated e^{-5t}).
3. A pole on the real axis generates an *exponential* response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero (i.e., again the pole at -5 generated e^{-5t}; see Figure 4.2 for the general case).
4. The zeros and poles generate the *amplitudes* for both the forced and natural responses (this can be seen from the calculation of A and B in Eq. (4.1)).

Let us now look at an example that demonstrates the technique of using poles to obtain the form of the system response. We will learn to write the form of the response by inspection. Each pole of the system transfer function that is on the real axis generates an exponential response that is a component of the natural response. The input pole generates the forced response.

Figure 4.2
Effect of a real-axis pole upon transient response



Evaluating response using poles

Problem Given the system of Figure 4.3, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

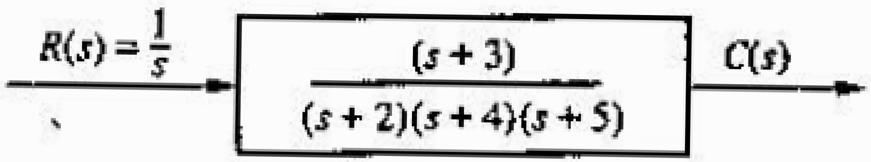


Figure 4.3 System for Example 4.1

Solution By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) = \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}}_{\text{Natural response}} \tag{4.3}$$

Taking the inverse Laplace transform, we get

$$c(t) = \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}} \tag{4.4}$$

4.3 First-Order Systems

A first-order system without zeros can be described by the transfer function shown in Figure 4.4(a). If the input is a unit step, where $R(s) = 1/s$, the Laplace transform of the step response is $C(s)$, where

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)} \tag{4.5}$$

Taking the inverse transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at} \tag{4.6}$$

where the input pole at the origin generated the forced response $c_f(t) = 1$, and the system pole at $-a$, as shown in Figure 4.4(b), generated the natural response $c_n(t) = -e^{-at}$. Equation (4.6) is plotted in Figure 4.5.

Let us examine the significance of parameter a, the only parameter needed to describe the transient response. When $t = 1/a$,

$$e^{-at}\big|_{t=1/a} = e^{-1} = 0.37 \tag{4.7}$$

or

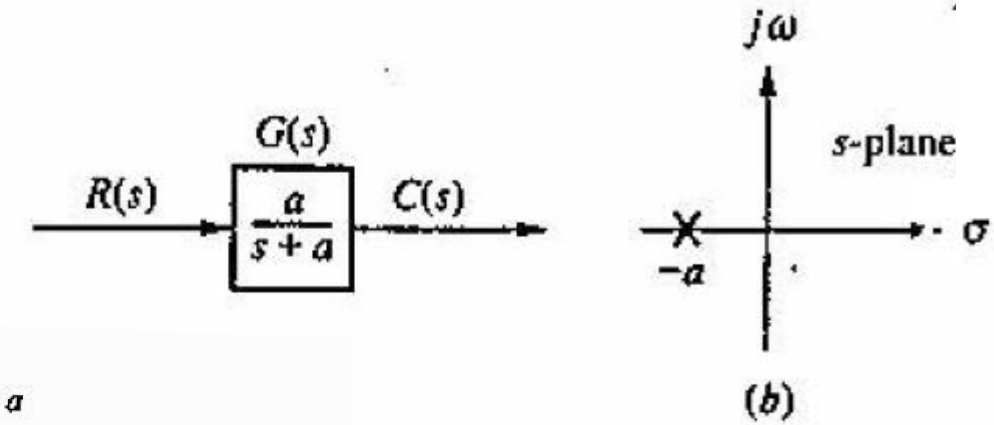
$$c(t)\big|_{t=1/a} = 1 - e^{-at}\big|_{t=1/a} = 1 - 0.37 = 0.63 \tag{4.8}$$

We now use Eqs. (4.6), (4.7), and (4.8) to define three transient response performance specifications.

Figure 4.4

a. First-order system;

b. pole plot



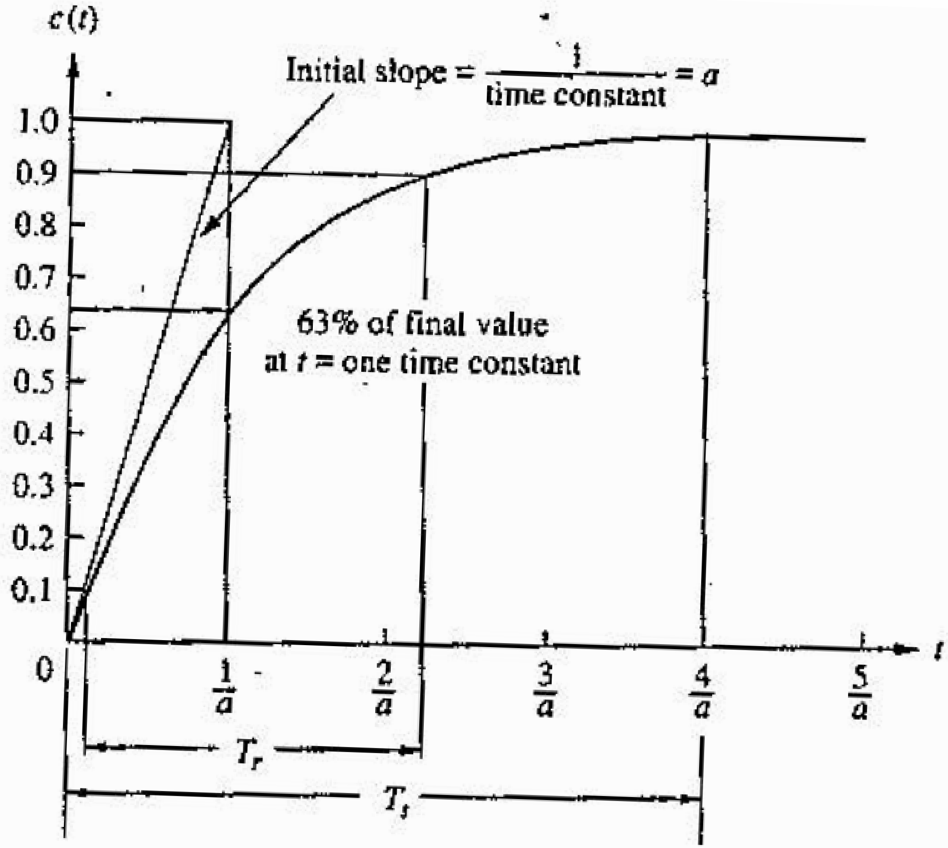


Figure 4.5

Time Constant

We call $1/a$ the *time constant* of the response. From Eq. (4.7), the time constant can be described as the time for e^{-at} to decay to 37% of its initial value. Alternately, from Eq. (4.8), the time constant is the time it takes for the step response to rise to 63% of its final value (see Figure 4.5).

The reciprocal of the time constant has the units (1/seconds), or frequency. Thus, we can call the parameter a the *exponential frequency.* Since the derivative of e^{-at} is $-a$ when $t = 0$, a is the initial rate of change of the exponential at $t = 0$. Thus, the time constant can be considered a transient response specification for a first-order system, since it is related to the speed at which the system responds to a step input.

The time constant can also be evaluated from the pole plot (see Figure 4.4(b)). Since the pole of the transfer function is at $-a$, we can say the pole is located at the *reciprocal* of the time constant, and the farther the pole from the imaginary axis, the faster the transient response.

Let us look at other transient response specifications such as *rise time*, T_r, and *settling time*, T_s, as shown in Figure 4.5.

Rise Time, T_r

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value. Rise time is found by solving Eq. (4.6) for the difference in time at $c(t) = 0.9$ and $c(t) = 0.1$. Hence,

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a} \tag{4.9}$$

Settling Time, T_s

Settling time is defined as the time for the response to reach, and stay within, 2% of its final value.[2] Letting $c(t) = 0.98$ in Eq. (4.6) and solving for time, t, we find the settling time to be

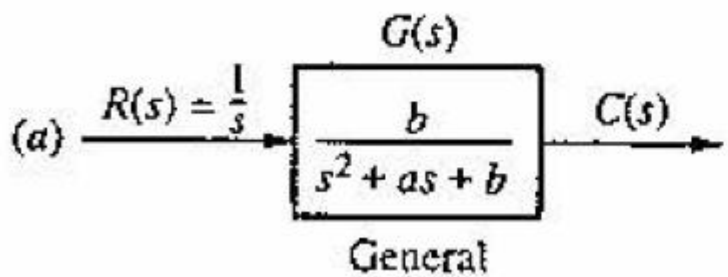
$$T_s = \frac{4}{a} \tag{4.10}$$

4.4 Second-Order Systems: Introduction

Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described. Whereas varying a first-order system's parameter simply changes the speed of the response, changes in the parameters of a second-order system can change the *form* of the response. For example, a second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its transient response.

To become familiar with the wide range of responses before formalizing our discussion in the next section, we take a look at numerical examples of the second-order system responses shown in Figure 4.7. All examples are derived from Figure 4.7(a), the general case, which has two finite poles and no zeros. The term in the numerator is simply a scale or input multiplying factor that can take on any value without affecting the form of the derived results. By assigning appropriate values to parameters a and b, we can show all possible second-order transient responses. The unit step response then can be found using $C(s) = R(s)G(s)$, where $R(s) = 1/s$, followed by a partial-fraction expansion and the inverse Laplace transform.

Figure 4.7



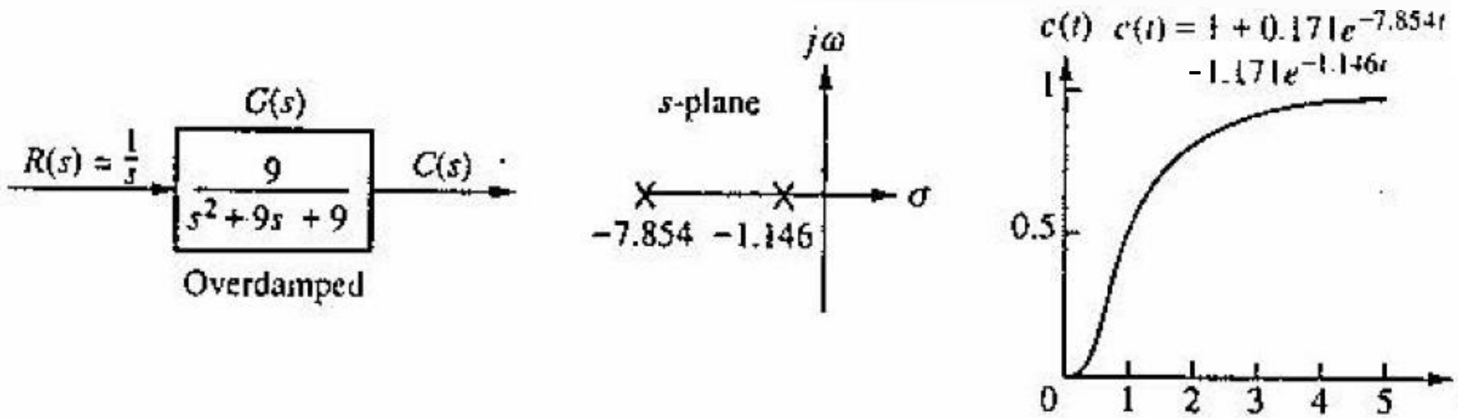
Overdamped Response, Figure 4.7(b)

For this response,

$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)} \qquad (4.12)$$

This function has a pole at the origin that comes from the unit step input and two real poles that come from the system. The input pole at the origin generates the constant forced response; each of the two system poles on the real axis generates an exponential natural response whose exponential frequency is equal to the pole location. Hence, the output initially could have been written as $c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$. This response, shown in Figure 4.7(b), is called *overdamped*.[3] We see that the poles tell us the form of the response without the tedious calculation of the inverse Laplace transform.

Figure 4.7 (b)



Underdamped Response, Figure 4.7(c)

For this response,

$$C(s) = \frac{9}{s(s^2 + 2s + 9)} \tag{4.13}$$

This function has a pole at the origin that comes from the unit step input and two complex poles that come from the system. We now compare the response of the second-order system to the poles that generated it. First we will compare the pole location to the time function, and then we will compare the pole location to the plot. From Figure 4.7(*c*), the poles that generate the natural response are at $s = -1 \pm j\sqrt{8}$. Comparing these values to $c(t)$ in the same figure, we see that the real part of the pole matches the exponential decay frequency of the sinusoid's amplitude, while the imaginary part of the pole matches the frequency of the sinusoidal oscillation.

Let us now compare the pole location to the plot. Figure 4.8 shows a general, damped sinusoidal response for a second-order system. The transient response consists of an exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole. The time constant of the exponential decay is equal to the reciprocal of the real part of the system pole. The value of the imaginary part is the actual frequency of the sinusoid, as depicted in Figure 4.8. This sinusoidal frequency is given the name *damped frequency of oscillation*, ω_d. Finally, the steady-state response (unit step) was generated by the input pole located at the origin. We call the type of response shown in Figure 4.8 an *underdamped response*, one which approaches a steady-state value via a transient response that is a damped oscillation.

Figure 4.7

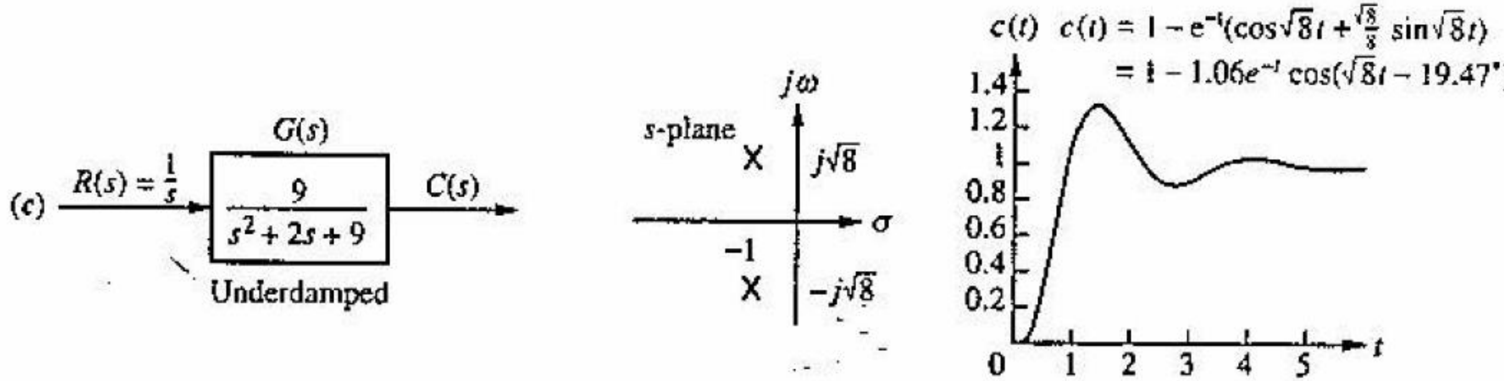
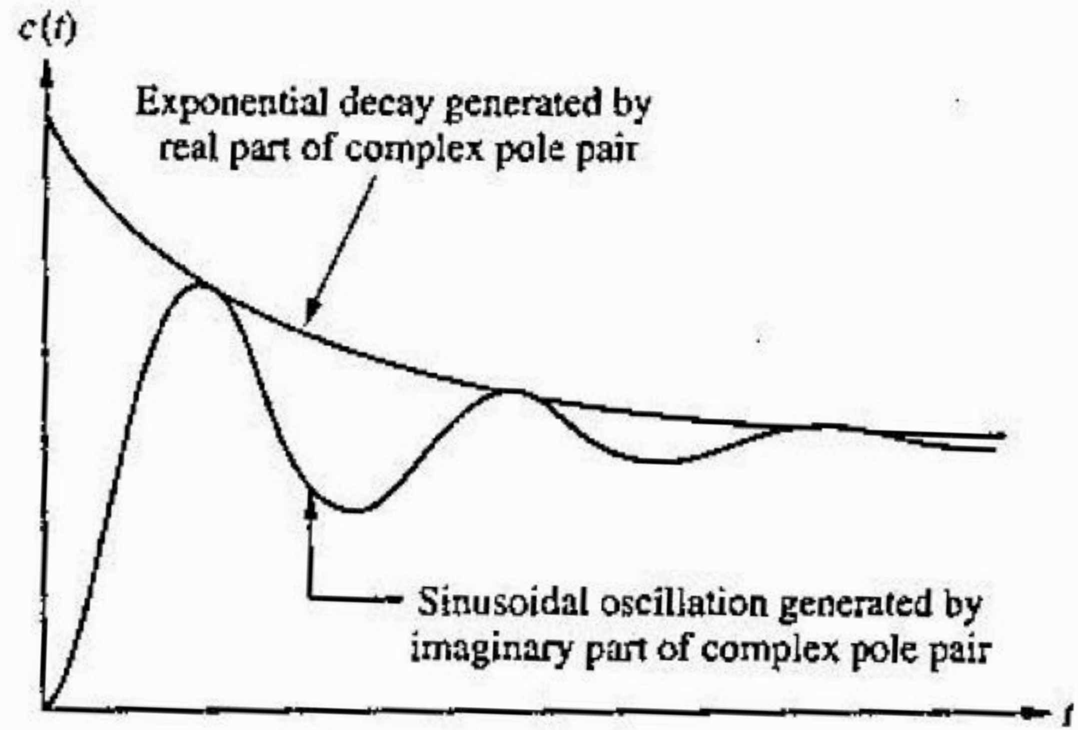


Figure 4.8



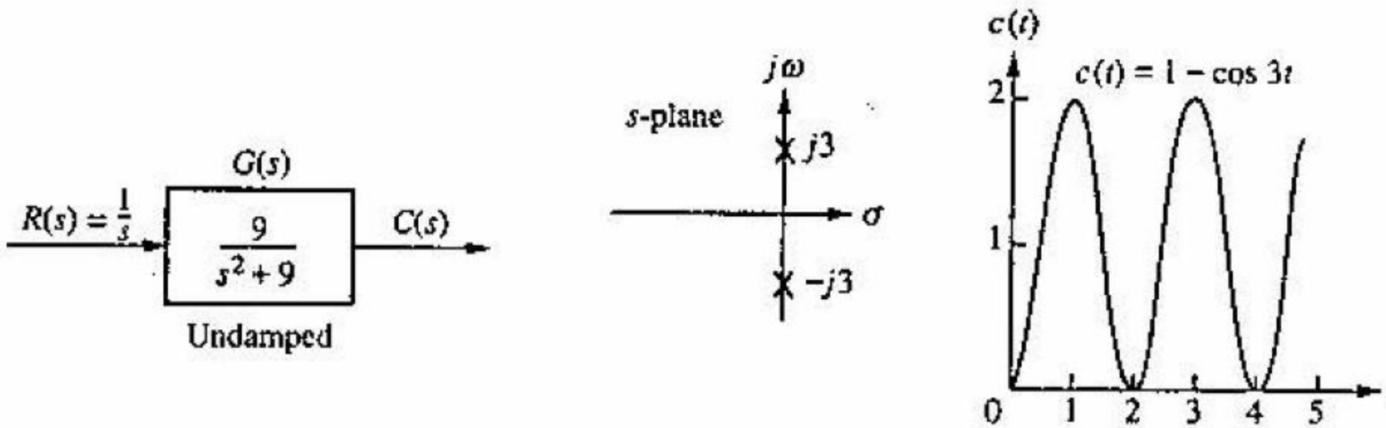
Undamped Response, Figure 4.7(*d*)

For this response,

$$C(s) = \frac{9}{s(s^2+9)} \tag{4.14}$$

This function has a pole at the origin that comes from the unit step input and two imaginary poles that come from the system. The input pole at the origin generates the constant forced response, and the two system poles on the imaginary axis at $\pm j3$ generate a sinusoidal natural response whose frequency is equal to the location of the imaginary poles. Hence, the output can be estimated as $c(t) = K_1 + K_4 \cos(3t - \phi)$. This type of response, shown in Figure 4.7(*d*), is called *undamped*. Note that the absence of a real part in the pole pair corresponds to an exponential that does not decay. Mathematically the exponential is $e^{-0t} = 1$.

Figure 4.7 (*d*)



1. *Overdamped responses:*

Poles: Two real at $-\sigma_1, -\sigma_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses:*

Poles: Two complex at $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t} \cos(\omega_d t - \phi)$$

3. *Undamped responses:*

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$

4.5 The General Second-Order System

Now that we have become familiar with second-order systems and their responses, we generalize the discussion and establish quantitative specifications defined in such a way that the response of a second-order system can be described to a designer without the need for sketching the response. In this section we define two physically meaningful specifications for second-order systems. These quantities can be used to describe the characteristics of the second-order transient response just as time constants describe the first-order system response. The two quantities are called *natural frequency* and *damping ratio*. Let us formally define them.

Natural Frequency, ω_n

The natural frequency of a second-order system is the frequency of oscillation of the system without damping. For example, the frequency of oscillation of a series *RLC* circuit with the resistance shorted would be the natural frequency.

Damping Ratio, ζ

Before we state our next definition, some explanation is in order. We have already seen that a second-order system's underdamped step response is characterized by damped oscillations. Our next definition is derived from the need to quantitatively describe this damped oscillation regardless of the time scale. Thus, a system whose transient response goes through three cycles in a millisecond before reaching the steady state would have the same measure as a system that went through three cycles in a millennium before reaching the steady state. For example, the underdamped curve in Figure 4.10 has an associated measure that defines its shape. This measure remains the same even if we change the time base from seconds to microseconds or to millennia.

A viable definition for this quantity is one that compares the exponential decay frequency of the envelope to the natural frequency. This ratio is constant regardless of the time scale of the response. Also, the reciprocal, which is proportional to the ratio of the natural period to the exponential time constant, remains the same regardless of the time base.

We define the damping ratio, ζ, to be

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}$$

Let us now revise our description of the second-order system to reflect the new definitions. The general second-order system shown in Figure 4.7(*a*) can be transformed to show the quantities ζ and ω_n. Consider the general system

$$G(s) = \frac{b}{s^2 + as + b} \tag{4.16}$$

Without damping, the poles would be on the $j\omega$ axis, and the response would be an undamped sinusoid. For the poles to be purely imaginary, $a = 0$. Hence,

$$G(s) = \frac{b}{s^2 + b} \tag{4.17}$$

By definition, the natural frequency, ω_n, is the frequency of oscillation of this system. Since the poles of this system are on the $j\omega$ axis at $\pm j\sqrt{b}$,

$$\omega_n = \sqrt{b} \tag{4.18}$$

Hence,

$$b = \omega_n^2 \tag{4.19}$$

Now what is the term a in Eq. (4.16)? Assuming an underdamped system, the complex poles have a real part, σ, equal to $-a/2$. The magnitude of this value is then the exponential decay frequency described in Section 4.4. Hence,

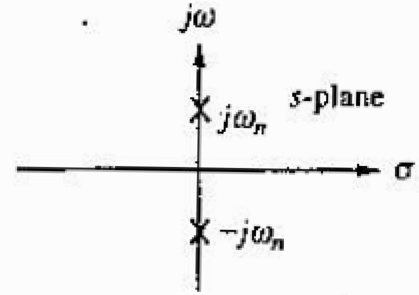
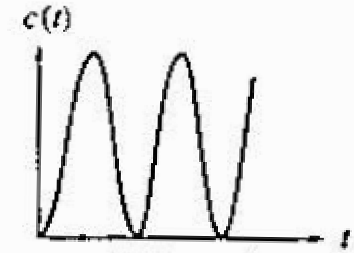
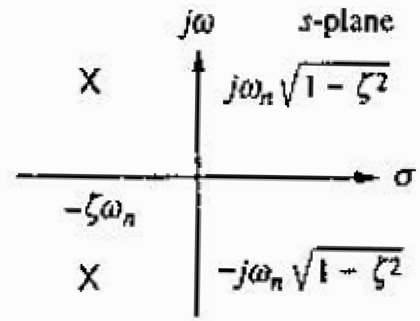
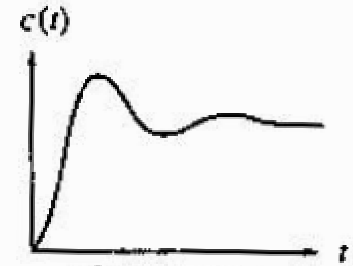
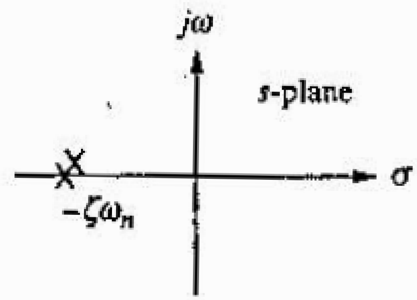
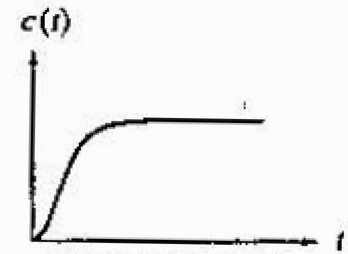
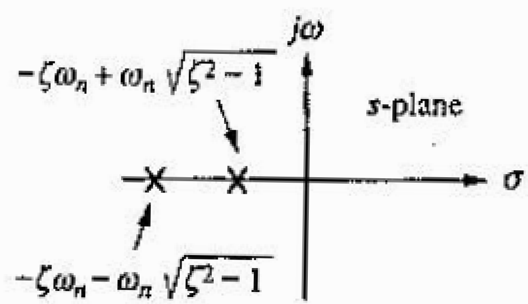
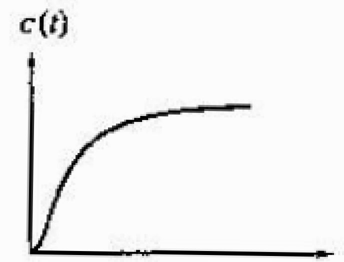
$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n} \tag{4.20}$$

from which

$$a = 2\zeta\omega_n \tag{4.21}$$

Our general second-order transfer function finally looks like this:

$$G(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \tag{4.22}$$

ζ	Poles	Step response
0		 Undamped
$0 < \zeta < 1$		 Underdamped
$\zeta = 1$		 Critically damped
$\zeta > 1$		 Overdamped

4.6 Underdamped Second-Order Systems

The underdamped second-order system, a common model for physical problems, displays unique behavior that must be itemized; a detailed description of the underdamped response is necessary for both analysis and design. Our first objective is to define transient specifications associated with underdamped responses. Next, we relate these specifications to the pole location, drawing an association between pole location and the form of the underdamped second-order response. Finally, we tie the pole location to system parameters, thus closing the loop: desired response generates required system components.

Let us begin by finding the step response for the general second-order system of Eq. (4.22). The transform of the response, $C(s)$, is the transform of the input times the transfer function, or

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.26}$$

where it is assumed that $\zeta < 1$ (i.e., the underdamped case). Expanding by partial fractions, using the methods described in Section 2.2, Case 3, yields

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \tag{4.27}$$

Taking the inverse Laplace transform, which is left as an exercise for the student, produces

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n \sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\,t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}\,t - \phi\right) \tag{4.28}$$

where $\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$.

A plot of this response appears in Figure 4.13 for various values of ζ, plotted along a time axis normalized to the natural frequency. We now see the relationship between the value of ζ and the type of response obtained: The lower the value of ζ, the more oscillatory the response. The natural frequency is a time-axis scale factor and does not affect the nature of the response other than to scale it in time.

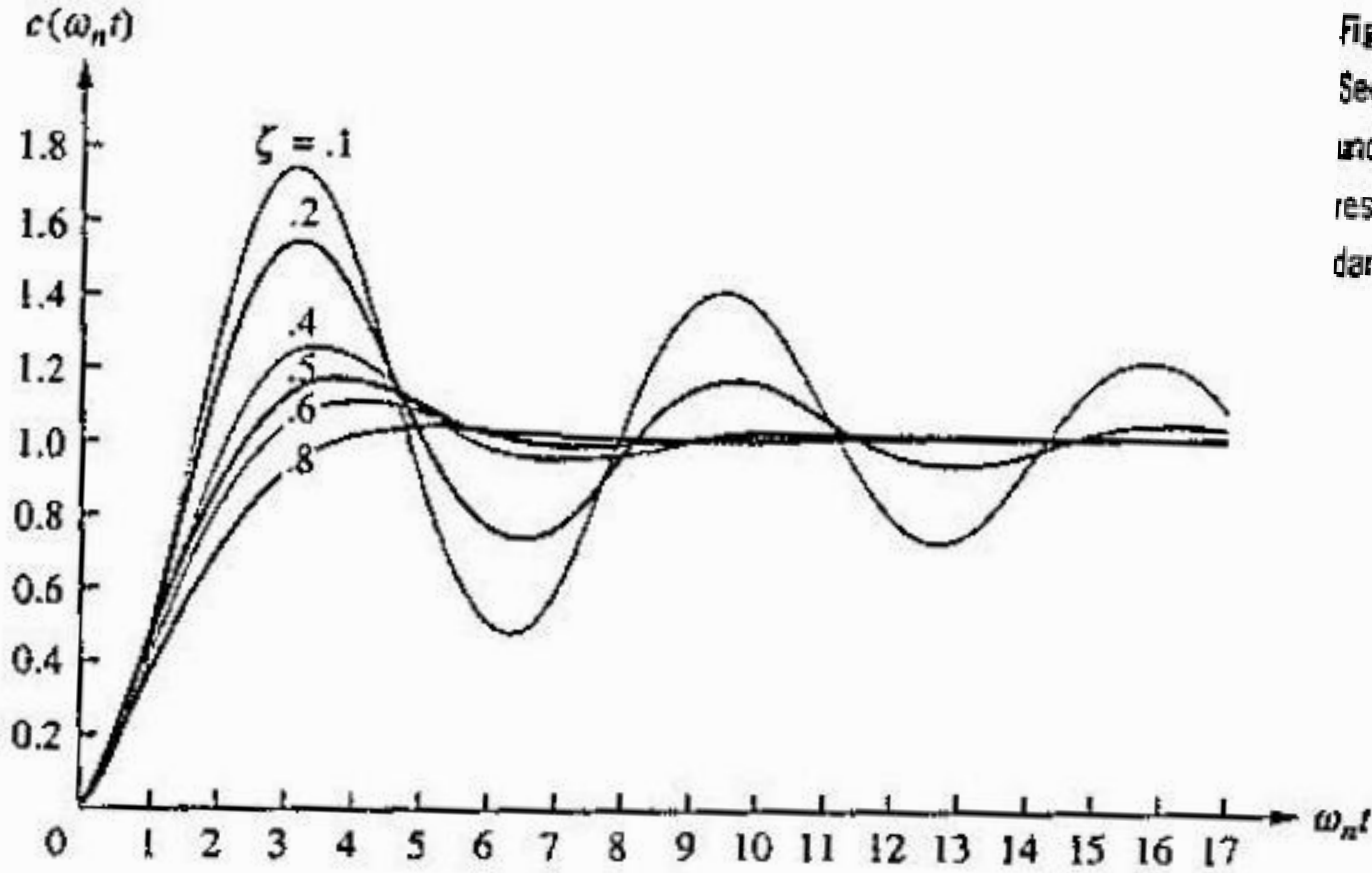
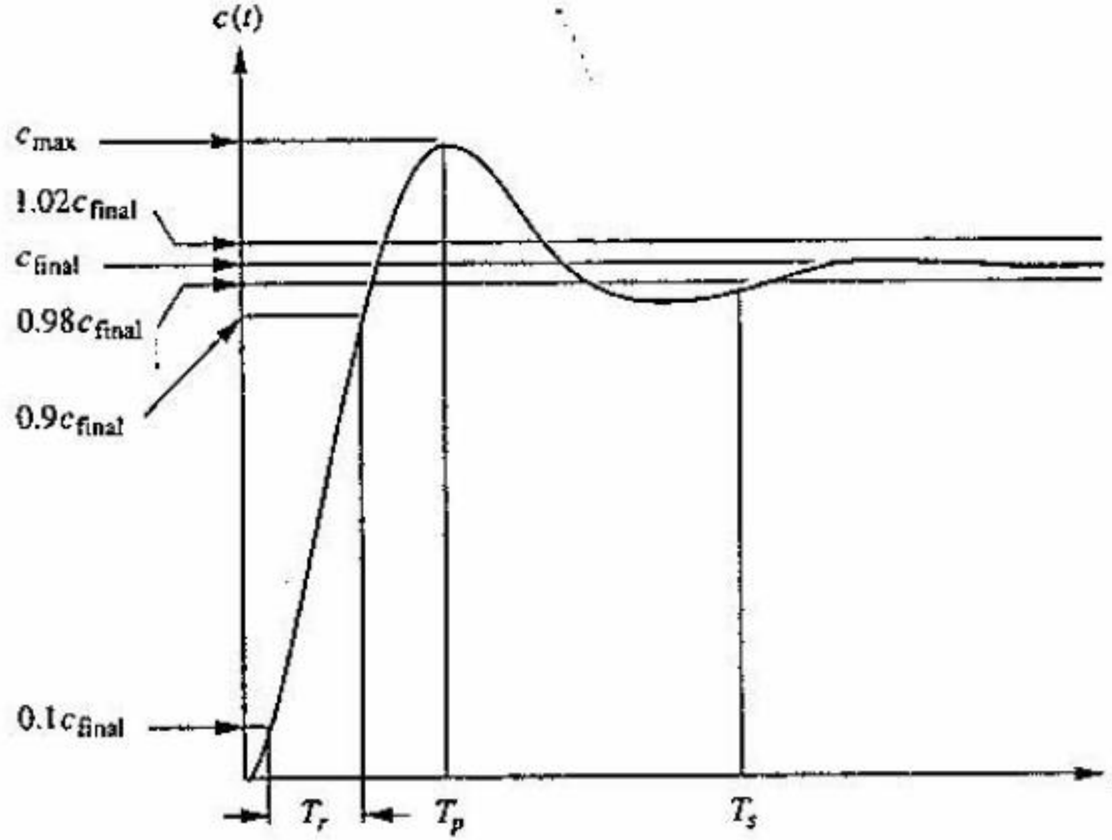


Figure 4.13 Second-order underdamped responses for damping ratio values

Figure 4.14 Second-order underdamped response specifications



1. *Peak time, T_p:* The time required to reach the first, or maximum, peak.
2. *Percent overshoot, %OS:* The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
3. *Settling time, T_s:* The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.
4. *Rise time, T_r:* The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

Notice that the definitions for settling time and rise time are basically the same as the definitions for the first-order response. All definitions are also valid for systems of order higher than 2, although analytical expressions for these parameters cannot be found unless the response of the higher-order system can be approximated as a second-order system.

For a second-order system, step response:

Evaluation of T_p

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

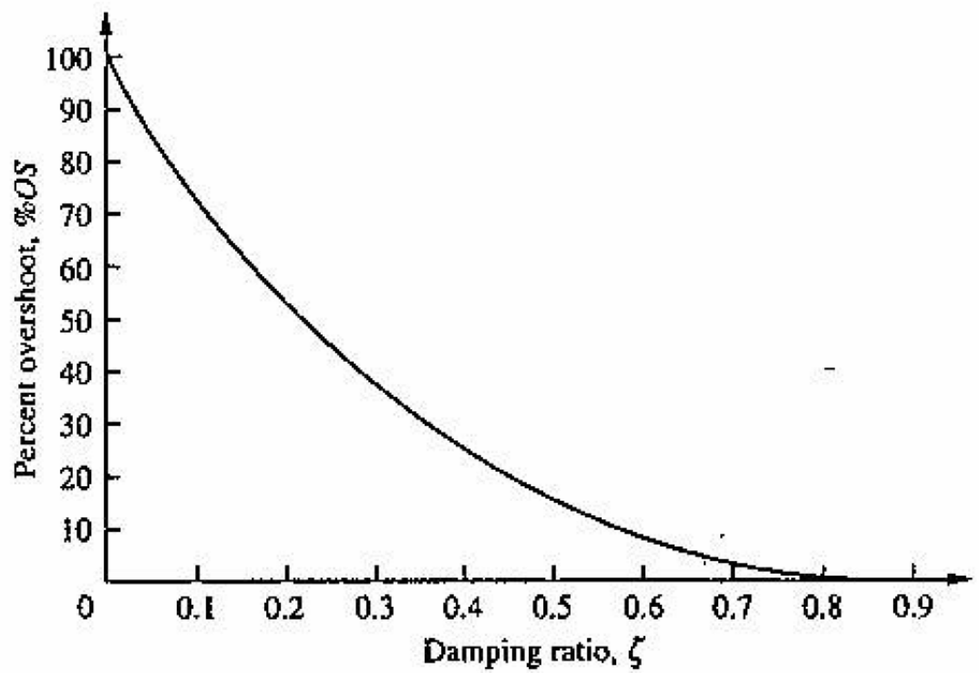
Evaluation of %OS

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

The inverse is given by

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



Evaluation of T_s

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \tag{4.41}$$

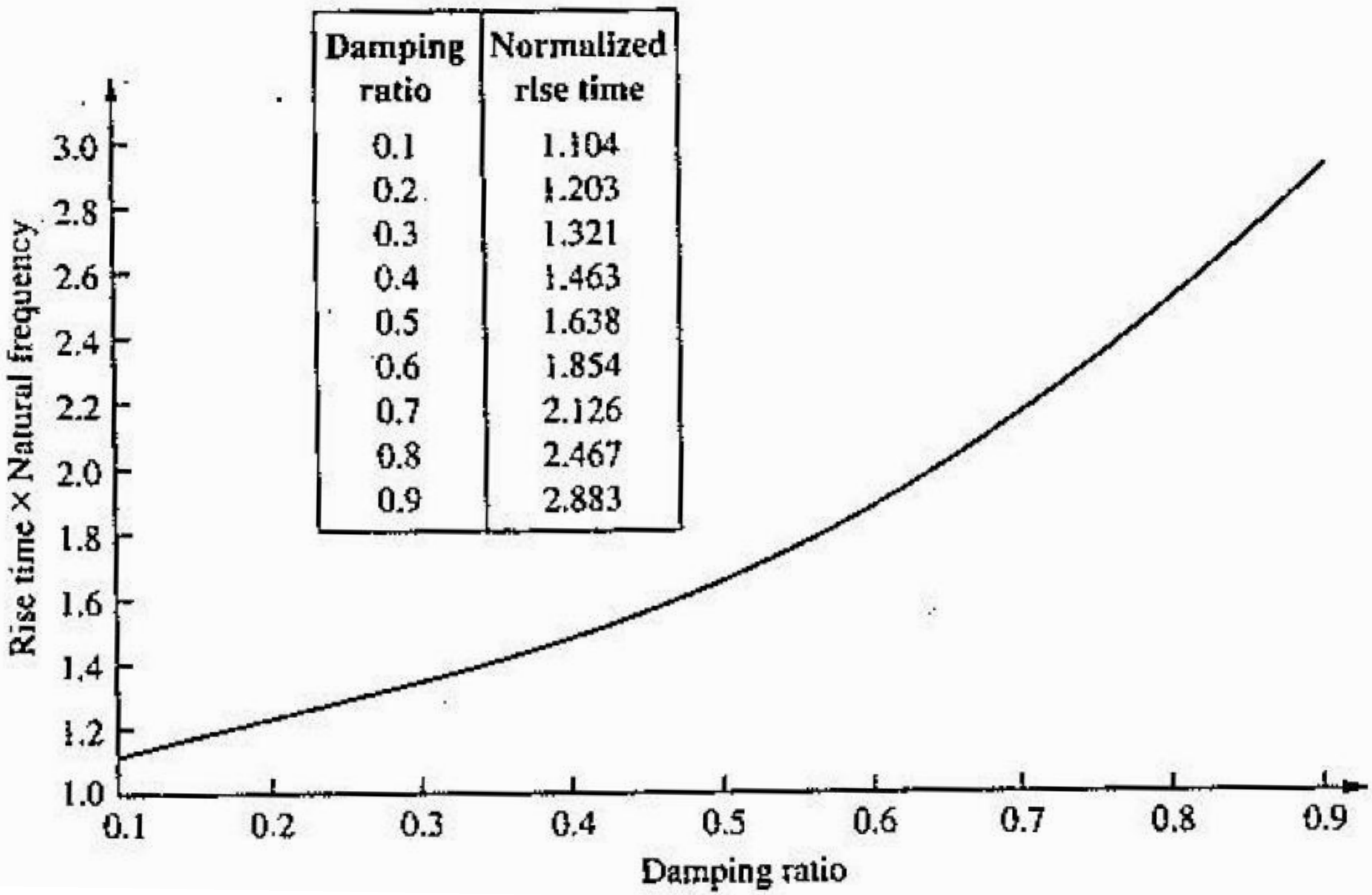
You can verify that the numerator of Eq. (4.41) varies from 3.91 to 4.74 as ζ varies from 0 to 0.9. Let us agree on an approximation for the settling time that will be used for all values of ζ; let it be

$$T_s = \frac{4}{\zeta\omega_n}$$

Evaluation of T_r

A precise analytical relationship between rise time and damping ratio, ζ, cannot be found. However, using a computer and Eq. (4.28), the rise time can be found. We first designate $\omega_n t$ as the normalized time variable and select a value for ζ. Using the computer, we solve for the values of $\omega_n t$ that yield $c(t) = 0.9$ and $c(t) = 0.1$. Subtracting the two values of $\omega_n t$ yields the normalized rise time, $\omega_n T_r$, for that value of ζ. Continuing in like fashion with other values of ζ, we obtain the results plotted in Figure 4.16

Figure 4.16



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

4.7 System Response with Additional Poles

In the last section we analyzed systems with one or two poles. It must be emphasized that the formulae describing percent overshoot, settling time, and peak time were derived only for a system with two complex poles and no zeros. If a system such as that shown in Figure 4.22 has more than two poles or has zeros, we cannot use the formulae to calculate the performance specifications that we derived. However, under certain conditions, a system with more than two poles or with zeros can be approximated as a second-order system that has just two complex *dominant poles*. Once we justify this approximation, the formulae for percent overshoot, settling time, and peak time can be applied to these higher-order systems using the location of the dominant poles. In this section we investigate the effect of an additional pole on the second-order response.

Problem Find the step response of each of the transfer functions shown in Eqs. (4.62) through (4.64) and compare them.

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \tag{4.62}$$

$$T_2(s) = \frac{245.42}{(s + 10)(s^2 + 4s + 24.542)} \tag{4.63}$$

$$T_3(s) = \frac{73.626}{(s + 3)(s^2 + 4s + 24.542)} \tag{4.64}$$

Solution The step response, $C_i(s)$, for the transfer function, $T_i(s)$, can be found by multiplying the transfer function by $1/s$, a step input, and using partial-fraction expansion followed by the inverse Laplace transform to find the response, $c_i(t)$. With the details left as an exercise for the student, the results are

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8°) \tag{4.65}$$

$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°) \tag{4.66}$$

$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°) \tag{4.67}$$

The three responses are plotted in Figure 4.24. Notice that $c_2(t)$, with its third pole at -10 and farthest from the dominant poles, is the better approximation of $c_1(t)$, the pure second-order system response; $c_3(t)$, with a third pole close to the dominant poles, yields the most error.